Single flavour pseudofermions on the lattice

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QCD Down Under
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In Lattice QCD, we generate configurations with quarks fields $\psi^{(f)}$ on the lattice sites and gluon fields $U_\mu$ on the lattice links.

Many simulations combine the up and the down quark into a single entity $n_f = 2$ for convenience.
Recent lattice simulations add in QED effects, which is done by introducing a photon field $A^\text{QED}_\mu$.

But now the up and down quark are distinguished by their charge, so we can no longer use the $n_f = 2$ trick.

We must use **single flavour pseudofermions** $n_f = 1$. 

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**Lattice QCD+QED**
Outline

1. Theory: what are single flavour pseudofermions?

2. Results: how fast can we generate them?
   - Filtering methods
   - Multiple filters
   - Characteristic scale tuning
Lattice gauge theory centres around the evaluation of expectation values of operators $O$:

$$
\langle O \rangle = \frac{\int DUD\psi D\bar{\psi} O[U, \psi^{(f)}, \bar{\psi}^{(f)}] e^{-S[U, \psi^{(f)}, \bar{\psi}^{(f)}]}}{\int DUD\psi D\bar{\psi} e^{-S[U, \psi^{(f)}, \bar{\psi}^{(f)}]}}
$$

where the lattice action $S$ is defined as

$$
S[U, \psi^{(f)}, \bar{\psi}^{(f)}] = S_G[U] + S'_F[U, \psi^{(f)}, \bar{\psi}^{(f)}] = S_G[U] + \sum_f \bar{\psi}^{(f)} M^{(f)} \psi^{(f)}
$$

with $M^{(f)}$ being the Dirac operator for fermion flavour $f$. 

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Expectation values

- We evaluate expectation values on the lattice by generating configurations \((U_i, \psi_i^{(f)}, \bar{\psi}_i^{(f)})\) distributed according to the Boltzmann distribution \(\frac{1}{Z} \exp(-S[U, \psi^{(f)}, \bar{\psi}^{(f)}])\), then taking the average

\[
\langle O \rangle \approx \frac{1}{N} \sum_{i=1}^{N} O[U_i, \psi_i^{(f)}, \bar{\psi}_i^{(f)}].
\]

- The effort lies in generating appropriate configurations, for which we use Hybrid Monte Carlo (HMC).
Double flavour pseudo-fermions

- HMC requires the fermions $\psi, \bar{\psi}$ to be in a non-Grassmannian form.
- Consider the fermion part of the partition function $Z$ for an action with degenerate up and down quarks

$$\int D\psi D\bar{\psi} \exp \left[ -\bar{\psi}_u M_u \psi_u - \bar{\psi}_d M_d \psi_d \right]$$

$$= \det M_u \det M_d$$

$$= \det M^\dagger M$$

$$= \int D\phi D\phi^* \exp \left[ -\phi^\dagger (M^\dagger M)^{-1} \phi \right]$$

- This is now a regular integral with pseudofermions $\phi$, and hence we can generate appropriate configurations via HMC with the lattice fermion action $S_F = \phi^\dagger (M^\dagger M)^{-1} \phi$. 
Hybrid Monte Carlo

Goal

Generate configurations \((U_i, \phi_i)\) distributed according to \(\frac{1}{Z} e^{-S[U,\phi]}\).

- Generate \(\phi\) from the Gaussian distribution \(\exp(-S_F[U,\phi])\)
- Introduce a “momentum” field \(P\) conjugate to the gauge field \(U\), and construct the Hamiltonian

\[
H = \sum \text{Tr} P^2 + S[U,\phi].
\]
Hybrid Monte Carlo

Goal

Generate configurations \((U_i, \phi_i)\) distributed according to \(\frac{1}{Z} e^{-S[U,\phi]}\).

- Integrate a molecular dynamics trajectory based on Hamilton’s equations, i.e. using steps
  \[
  (P, U) \rightarrow (P, e^{ihP}U) \quad \text{and} \quad (P, U) \rightarrow (P - hF, U),
  \]
  where \(F = \frac{\partial S}{\partial U}\) is the force term. This produces the candidate state \((P', U')\).
- Accept this state with probability \(\min(1, \exp[H - H'])\).
Consider an action with just the strange quark. The fermion part of the partition function can be written as

$$\text{det } M_s = \int D\phi D\phi^* \exp \left[ -\phi^+ M_s^{-1} \phi \right]$$
Single-flavour pseudo-fermions?

- Consider an action with just the strange quark. The fermion part of the partition function can be written as

\[
\text{det } M_s = \int D\phi D\phi^* \exp \left[ -\phi^\dagger M_s^{-1} \phi \right]
\]

Problem

The pseudo-fermion integral is not guaranteed to converge, as \( M_s \) may have negative eigenvalues.

- This is not a problem with the double-flavour pseudo-fermion because \( M^\dagger M \) is positive semi-definite by construction.
Rational Hybrid Monte Carlo

Idea
Replace the matrix $M_s^{-1}$ with a rational approximation $R(K) \approx K^{-1/2}$ where $K = M_s^\dagger M_s$.

- A rational function of $M_s^\dagger M_s$ is positive semi-definite by construction, and so we can safely use HMC with fermion action

\[ S_F = \phi^\dagger R(K) \phi. \]

- This technique of replacing the fermion action with a rational approximation is known as Rational Hybrid Monte Carlo (RHMC).
Rational Hybrid Monte Carlo

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Replace the matrix $M_s^{-1}$ with a rational approximation $R(K) \approx K^{-1/2}$ where $K = M_s^\dagger M_s$.

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$$S_F = \phi^\dagger R(K) \phi.$$  

- This technique of replacing the fermion action with a rational approximation is known as Rational Hybrid Monte Carlo (RHMC).

Can we do better?
Filtering methods

One way to reduce the cost of HMC is to split the action into multiple terms via a filter $F$ on the fermion action kernel $L$

$$\phi^\dagger L\phi \rightarrow \phi_1^\dagger F\phi_1 + \phi_2^\dagger F^{-1}L\phi_2,$$

then integrate each term with a different step-size $h_i$. 

\[ L \quad F \quad F^{-1}L \]
Single flavour filters: Polynomial filtered RHMC

- Consider filtering the RHMC action
  \[ S_{RHMC} = \phi^\dagger R(K) \phi \]

- One choice of filter is a low order polynomial \( P(K) \) that approximates \( K^{-1/2} \), giving fermion action:
  \[ S_{PF-RHMC} = \phi_1^\dagger P(K) \phi_1 + \phi_2^\dagger P(K)^{-1} R(K) \phi_2. \]

- By choosing a particular class of polynomials \( P(K) \), we only need to tune the polynomial order \( p \).
Single flavour filters: tRHMC

- We can write the rational approximation $R(K)$ as

$$R(K) = d_n \prod_{k=1}^{n} \frac{(K + a_k)}{(K + b_k)}$$

where $a_k, b_k, d_n > 0$ and $a_{k+1} \leq a_k, b_{k+1} \leq b_k$.

- Assuming that the truncated ordered product given by

$$R_{i,j}(K) = (d_n)^{\delta_{i1}} \prod_{k=i}^{j} \frac{(K + a_k)}{(K + b_k)}$$

approximates $K^{-1/2}$ whenever $i = 1$, we can use $R_{1,t}(K)$ as a filter:

$$S_{tRHMC} = \phi_1^{\dagger} R_{1,t}(K) \phi_1 + \phi_2^{\dagger} R_{t+1,n}(K) \phi_2.$$
Lattice setup

- All runs were performed using BQCD, which has these filtering methods built into the latest release:
  www.rrz.uni-hamburg.de/bqcd

- The test lattice used in this work was a $16^3 \times 32$ lattice with Wilson gauge $\beta = 5.6$ and two degenerate single-flavour Wilson fermions at $\kappa = 0.15825$.

- This corresponds to lattice spacing $a \sim 0.08$ fm and pion mass $m_\pi \sim 400$ MeV.

Goal
Minimize the cost to generate independent configurations, which we parametrize as

$$\text{Cost} \propto \frac{N_{\text{mat}}}{P_{\text{acc}}}$$
Tuning the step-sizes

- Our actions have several step-sizes to tune; e.g. the PF-RHMC action

\[ S = S_G[U] + \phi_1^\dagger P(K)\phi_1 + \phi_2^\dagger P(K)^{-1}R(K)\phi_2 \]

has step-sizes \( \{h_G, h_1, h_2\} \).

- We tune these by force balancing, where given the forces \( F_i = \frac{\partial S_i}{\partial U} \) for each action term, the step-sizes follow

\[ F_i h_i \approx \text{constant} \]

- We then tune the only free step-size, the coarsest scale \( h_n \), such that the acceptance rate \( P_{acc} \sim 0.75 \), which can be performed by fitting \( P_{acc} \) to

\[ P_{acc} \approx \text{erfc}(h_n^2c^2). \]
Filtering: Results

The graph illustrates the filtering methods for different cost (R, PF, tRHMC) and parameters (p, t) as follows:

- **R**: Represents the filtering results for a fixed parameter p = 4.
- **PF**: Shows the filtering results for a fixed parameter t1 = 4.
- **tRHMC**: Displays the filtering results for a fixed parameter t = 4.

The data points are plotted with error bars indicating the cost values for various combinations of p and t, as well as q and t2, with specific values indicated by the parameters within the graph.

This visualization is crucial for understanding the performance and efficiency of different filtering methods under various conditions.
Combining filters

We can improve the performance by using two filters.

2PF-RHMC

\[ S_{2PF-RHMC} = \phi_1^\dagger P(K)\phi_1 + \phi_2^\dagger Q(K)\phi_2 + \phi_3^\dagger R(K)[P(K)Q(K)]^{-1}\phi_3 \]

where \( Q(K) \approx P(K)^{-1}K^{-1/2} \) is a polynomial \( q > p \)

2tRHMC

\[ S_{2tRHMC} = \phi_1^\dagger R_{1,t_1}(K)\phi_1 + \phi_2^\dagger R_{t_1+1,t_2}(K)\phi_2 + \phi_3^\dagger R_{t_2+1,n}(K)\phi_3 \]

PF-tRHMC

\[ S_{PF-tRHMC} = \phi_1^\dagger P(K)\phi_1 + \phi_2^\dagger R_{1,t}(K)P(K)^{-1}\phi_2 + \phi_3^\dagger R_{t+1,n}(K)\phi_3 \]
Combining filters: Results

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Combining filters: Results

![Graph showing the results of combining filters]

- Cost
- Filters: R, PF, tRHMC, 2PF, 2tRHMC, PF-tRHMC
- Parameters: p, t, q, t2, t
- Single flavour pseudofermions
- QCD Down Under '17
Characteristic scale tuning

- Looking at some of the previous results (e.g. the $t = 2, 8$ points for 1tRHMC), it seems that force balancing does not perform well for all choices of filter.
- The idea of **characteristic scale tuning** is to tune the coarsest scale via fitting to $\text{erfc}(h_n^2c^2)$ as before, then **tune the second-coarsest scale manually** while keeping the remaining step-sizes in sync according to $F_ih_i \approx \text{constant}$.
- This can help offset an unfavourable distribution of forces between action terms.
CS-tuning: tRHMC

![Graph showing characteristic scale tuning for different processes and particle types.](image-url)
CS-tuning: tRHMC

Results of characteristic scale tuning for different scales and parameters.

<table>
<thead>
<tr>
<th>Scale</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td></td>
</tr>
<tr>
<td>PF</td>
<td></td>
</tr>
<tr>
<td>tRHMC</td>
<td></td>
</tr>
<tr>
<td>2PF</td>
<td></td>
</tr>
<tr>
<td>2tRHMC</td>
<td></td>
</tr>
<tr>
<td>PF-tRHMC</td>
<td></td>
</tr>
</tbody>
</table>

Parameters:
- \( p \) (flavours)
- \( t \) (temperature)
- \( q \) (\( p=4 \))
- \( t_2 \) (\( t_1=4 \))
- \( t \) (\( p=4 \))

Graph showing the cost for different scales and parameters with and without CS-tuning.
CS-tuning: PF-RHMC

![Graph showing the cost of different tunings for PF-RHMC with and without characteristic scale (CS) tuning. The graph has columns for R, PF, tRHMC, 2PF, 2tRHMC, and PF-tRHMC. The y-axis represents the cost, and the x-axis represents different parameters (p, t, q, t_2, t) with specific values.]
CS-tuning: PF-RHMC

The figure shows a scatter plot comparing the performance of different tuning methods: PF, RHMC, and their combinations. The x-axis represents different parameters (p, t, q, t2, t) and the y-axis represents cost. The red circles indicate the performance without characteristic scale tuning (No CS), and the blue circles indicate the performance with characteristic scale tuning (CS). The plot illustrates how tuning can impact the cost of computations in a pseudofermion framework, potentially reducing computational resources required for simulations.
Results

Characteristic scale tuning

CS-tuning: PF-tRHMC

![Graph showing results of CS-tuning for PF-tRHMC]

- **No CS**
- **CS**

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CS-tuning: PF-tRHMC

![Graph showing results of characteristic scale tuning for PF-tRHMC, with data points for cost and configurations.]

- **Results**
  - Characteristic scale tuning
  - **CS-tuning**: PF-tRHMC

- **Graph Details**
  - X-axis: $p$, $t$, $q$ (for $p=4$), $t_2$ (for $t_1=4$), $t$ (for $p=4$)
  - Y-axis: Cost
  - Data points for No CS and CS configurations

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We tried two different filtering methods to improve the cost of generating single-flavour pseudofermions: polynomial filtering and tRHMC. tRHMC performs the best, with about a $\times 1.5$ speedup. Characteristic scale tuning makes tRHMC work well across a wide range of truncation orders. tRHMC is simple to implement into existing RHMC codes. Multiple truncation filters can be used, should prove useful at light quark masses.
Backup slides
Full cost plot

![Graph showing the cost plot for different parameters.](image)

- **R**
- **PF**
- **tRHMC**
- **2PF**
- **2tRHMC**
- **PF-tRHMC**

- **Cost**

- **Parameters**:
  - **p**: single flavour pseudofermions
  - **t**: Single flavour pseudofermions
  - **q**: Single flavour pseudofermions
  - **t_2**: Single flavour pseudofermions
  - **t**: Single flavour pseudofermions

- **Legend**:
  - **No CS**
  - **CS**

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Full cost plot

The diagram shows a graph with the following axes:

- Cost on the y-axis
- Parameters $p$, $t$, $q$, $t_2$, and $t$ on the x-axis

The plot compares different methods and conditions, indicated by markers:

- Red markers represent No CS
- Blue markers represent CS

The graph is labeled with parameters:

- $p = 4$ for $q$
- $t_1 = 4$ for $t_2$
- $p = 4$ for $t$

The graph is used to analyze the costs associated with various computational methods and conditions.
## Cost data: PF-RHMC

<table>
<thead>
<tr>
<th>$p$</th>
<th>Steps</th>
<th>$N_{mat}$</th>
<th>$P_{acc}$</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>12-480</td>
<td>47,000(400)</td>
<td>0.69(3)</td>
<td>$6.8(4) \times 10^4$</td>
</tr>
<tr>
<td>4</td>
<td>10-18-480</td>
<td>42,200(300)</td>
<td>0.68(3)</td>
<td>62,000(3000)</td>
</tr>
<tr>
<td>10</td>
<td>10-26-480</td>
<td>47,200(500)</td>
<td>0.70(4)</td>
<td>67,000(5000)</td>
</tr>
<tr>
<td>16</td>
<td>11-35-480</td>
<td>51,700(300)</td>
<td>0.75(3)</td>
<td>69,000(3000)</td>
</tr>
</tbody>
</table>

**Table:** PF-RHMC config data (force tuned)
## Cost data: PF-RHMC

<table>
<thead>
<tr>
<th>$p$</th>
<th>Steps</th>
<th>$N_{\text{mat}}$</th>
<th>$P_{\text{acc}}$</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>12-480</td>
<td>47,000(400)</td>
<td>0.69(3)</td>
<td>$6.8(4) \times 10^4$</td>
</tr>
<tr>
<td>4</td>
<td>10-30-480</td>
<td>41,600(300)</td>
<td>0.71(3)</td>
<td>58,700(3000)</td>
</tr>
<tr>
<td>10</td>
<td>10-30-480</td>
<td>44,900(300)</td>
<td>0.73(4)</td>
<td>62,000(4000)</td>
</tr>
<tr>
<td>16</td>
<td>11-35-480</td>
<td>51,700(300)</td>
<td>0.75(3)</td>
<td>69,000(3000)</td>
</tr>
</tbody>
</table>

**Table:** PF-RHMC config data (c-scale tuned)
## Appendix

### Cost data: tRHMC

<table>
<thead>
<tr>
<th>$t$</th>
<th>Steps</th>
<th>$N_{mat}$</th>
<th>$P_{acc}$</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>15-5-480</td>
<td>57,400(500)</td>
<td>0.68(3)</td>
<td>85,000(5000)</td>
</tr>
<tr>
<td>3</td>
<td>10-10-480</td>
<td>40,000(400)</td>
<td>0.72(3)</td>
<td>56,000(3000)</td>
</tr>
<tr>
<td>4</td>
<td>8-20-480</td>
<td>34,700(200)</td>
<td>0.67(4)</td>
<td>52,000(3000)</td>
</tr>
<tr>
<td>5</td>
<td>6-35-480</td>
<td>34,200(300)</td>
<td>0.60(4)</td>
<td>57,000(5000)</td>
</tr>
<tr>
<td>6</td>
<td>5-52-480</td>
<td>40,800(200)</td>
<td>0.66(4)</td>
<td>62,000(5000)</td>
</tr>
<tr>
<td>8</td>
<td>4-110-480</td>
<td>124,400(300)</td>
<td>0.84(4)</td>
<td>148,000(7000)</td>
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</table>

**Table**: tRHMC config data (force tuned)
## Cost data: tRHMC

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<thead>
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<th>$t$</th>
<th>Steps</th>
<th>$N_{\text{mat}}$</th>
<th>$P_{\text{acc}}$</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>11-25-480</td>
<td>43,500(600)</td>
<td>0.71(3)</td>
<td>61,000(3000)</td>
</tr>
<tr>
<td>3</td>
<td>10-25-480</td>
<td>41,900(300)</td>
<td>0.76(3)</td>
<td>55,000(3000)</td>
</tr>
<tr>
<td>4</td>
<td>8-22-480</td>
<td>34,800(400)</td>
<td>0.76(3)</td>
<td>46,000(2000)</td>
</tr>
<tr>
<td>5</td>
<td>6-30-480</td>
<td>32,800(200)</td>
<td>0.70(3)</td>
<td>47,000(2000)</td>
</tr>
<tr>
<td>6</td>
<td>5-15-480</td>
<td>29,000(200)</td>
<td>0.71(3)</td>
<td>41,000(2000)</td>
</tr>
<tr>
<td>8</td>
<td>4-10-480</td>
<td>32,500(200)</td>
<td>0.68(3)</td>
<td>48,000(2000)</td>
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</table>

**Table:** tRHMC config data (c-scale tuned)
Cost data: 2PF-RHMC

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>Steps</th>
<th>$N_{mat}$</th>
<th>$P_{acc}$</th>
<th>Cost</th>
</tr>
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<tbody>
<tr>
<td>4</td>
<td>10</td>
<td>7-33-62-480</td>
<td>36,600(300)</td>
<td>0.66(3)</td>
<td>55,000(3000)</td>
</tr>
<tr>
<td>16</td>
<td>8-51-98-480</td>
<td>49,500(300)</td>
<td>0.67(4)</td>
<td>73,000(5000)</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>9-65-125-480</td>
<td>58,600(300)</td>
<td>0.71(4)</td>
<td>83,000(5000)</td>
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</tr>
</tbody>
</table>

Table: 2PF-RHMC config data (force tuned)
Cost data: 2PF-RHMC

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>Steps</th>
<th>$N_{\text{mat}}$</th>
<th>$P_{\text{acc}}$</th>
<th>Cost</th>
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</thead>
<tbody>
<tr>
<td>4</td>
<td>10</td>
<td>7-20-36-480</td>
<td>36,200(200)</td>
<td>0.67(3)</td>
<td>54,000(3000)</td>
</tr>
<tr>
<td>16</td>
<td>8-15-27-480</td>
<td>40,000(200)</td>
<td>0.65(3)</td>
<td>61,700(3000)</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>9-20-38-480</td>
<td>45,400(300)</td>
<td>0.72(3)</td>
<td>63,300(3000)</td>
<td></td>
</tr>
</tbody>
</table>

Table: 2PF-RHMC config data (c-scale tuned)
## Cost data: 2tRHMC

<table>
<thead>
<tr>
<th>$t_1$</th>
<th>$t_2$</th>
<th>Steps</th>
<th>$N_{mat}$</th>
<th>$P_{acc}$</th>
<th>Cost</th>
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<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>6-6-29-480</td>
<td>32,960(240)</td>
<td>0.67(4)</td>
<td>49,000(4000)</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>5-12-40-480</td>
<td>32,730(140)</td>
<td>0.72(3)</td>
<td>45,300(2100)</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>4-17-50-480</td>
<td>36,180(180)</td>
<td>0.71(3)</td>
<td>51,200(2400)</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>4-30-80-480</td>
<td>60,460(160)</td>
<td>0.85(3)</td>
<td>71,400(2600)</td>
</tr>
</tbody>
</table>

**Table:** 2tRHMC config data (force tuned)
## Cost data: 2tRHMC

<table>
<thead>
<tr>
<th>$t_1$</th>
<th>$t_2$</th>
<th>Steps</th>
<th>$N_{mat}$</th>
<th>$P_{acc}$</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>6-7-36-480</td>
<td>32,660(170)</td>
<td>0.68(4)</td>
<td>48,000(2400)</td>
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<tr>
<td>6</td>
<td>5</td>
<td>5-12-40-480</td>
<td>32,730(140)</td>
<td>0.72(3)</td>
<td>45,300(2100)</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>12-35-480</td>
<td>32,770(140)</td>
<td>0.69(3)</td>
<td>47,500(2300)</td>
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<tr>
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<td>4</td>
<td>10-30-480</td>
<td>35,080(190)</td>
<td>0.80(3)</td>
<td>44,000(1900)</td>
</tr>
</tbody>
</table>

**Table:** 2tRHMC config data (c-scale tuned)
## Cost data: PF-tRHMC

<table>
<thead>
<tr>
<th>$p$</th>
<th>$t$</th>
<th>Steps</th>
<th>$N_{mat}$</th>
<th>$P_{acc}$</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>8-4-27-480</td>
<td>31,000(300)</td>
<td>0.15(3)</td>
<td>210,000(50,000)</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>12-36-480</td>
<td>33,140(270)</td>
<td>0.60(4)</td>
<td>55,000(5000)</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>26-59-480</td>
<td>34,800(180)</td>
<td>0.75(3)</td>
<td>46,700(2400)</td>
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<tr>
<td>7</td>
<td>4</td>
<td>38-79-480</td>
<td>47,460(180)</td>
<td>0.71(4)</td>
<td>67,000(4000)</td>
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</table>

**Table:** PF-tRHMC config data (force tuned)
Cost data: PF-tRHMC

<table>
<thead>
<tr>
<th>$p$</th>
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<th>Steps</th>
<th>$N_{mat}$</th>
<th>$P_{acc}$</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
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<td>0.74(3)</td>
<td>50,100(2500)</td>
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<tr>
<td>5</td>
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<td>6-10-30-480</td>
<td>27,960(240)</td>
<td>0.68(3)</td>
<td>42,500(2200)</td>
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<tr>
<td>6</td>
<td>5</td>
<td>5-12-27-480</td>
<td>28,770(210)</td>
<td>0.69(3)</td>
<td>43,200(2200)</td>
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<td>4-16-32-480</td>
<td>32,640(180)</td>
<td>0.69(4)</td>
<td>49,000(3000)</td>
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*Table: PF-tRHMC config data (c-scale tuned)*
Force plots

RHMC  1PF-RHMC  1tRHMC

$F_1$

$F_2$

$p$

$t$
Appendix

Force plots

![Force plots graph]

- **2PF-RHMC**
- **2tRHMC**
- **PF-tRHMC**

Parameters:
- $F_1$, $F_2$, $F_3$
- $q$, $t_2$, $t$
## Force data

<table>
<thead>
<tr>
<th>$p$</th>
<th>$F_1$ avg</th>
<th>$F_1$ max</th>
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<th>$F_2$ max</th>
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<tbody>
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<td>–</td>
<td>1.69</td>
<td>4.93</td>
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<td>5.78</td>
<td>1.11</td>
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<td>1.82</td>
<td>5.15</td>
<td>0.65</td>
<td>1.62</td>
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*Table: PF-RHMC forces*
# Force data

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<td>3.72</td>
<td>0.453</td>
<td>1.43</td>
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<td>5</td>
<td>1.522</td>
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<td>4.72</td>
<td>0.0944</td>
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<tr>
<td>8</td>
<td>1.683</td>
<td>4.90</td>
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**Table:** tRHMC forces
## Force data

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<tbody>
<tr>
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<td>1.970</td>
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<td>1.147</td>
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<tr>
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<td>1.970</td>
<td>5.78</td>
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**Table:** 2PF-RHMC forces
## Force data

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<td>0.4387</td>
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*Table: 2tRHMC forces*
# Force data

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<th>$F_2$ avg</th>
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<td>2.764</td>
<td>0.0455</td>
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</tbody>
</table>

**Table**: PF-tRHMC forces
More lattice parameters

- The rational approximation used was the Zolotarev optimal rational approximation with order $n = 20$ and range $[5 \times 10^{-5}, 3]$.

- The polynomials used were Chebyshev approximations to $K^{-1/2}$ (or $P(K)^{-1}K^{-1/2}$ in the two filter case) with range $[5 \times 10^{-5}, 3]$. 